Adapted representations and applications

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Outline

Contents

September 9–12, 2014: Fouth edition of the famous MSTA (Modern Stochastics: Theory and Applications) conference series entitled "Fractality and Fractionality".

http://probability.univ.kiev.ua/msta4conf

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In the third edition, over 250 scientists from 29 countries took part, including such top-level specialists as Marc Yor, Leonid Pastur, Bernt Øksendal, Hanspeter Schmidli, Yuri Bakhtin, Hans-Jurgen Engelbert, Shizan Fang, Peter Imkeller, Enzo Orsingher, Pierre Vallois and others.

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Idea: since $\int_0^1 (1-s)^{-2} ds = \infty$, it holds $\underline{\lim}_{t \to 1-} \int_0^t (1-s)^{-1} dW_s = -\infty$ and $\overline{\lim}_{t \to 1-} \int_0^t (1-s)^{-1} dW_s = +\infty$. So, for example, to represent $\xi = 1$ as stochastic integral, let $v_t = \int_0^t (1-s)^{-1} dW_s$, $\tau = \inf\{t : v_t = 1\}$ and put $\phi_t = (1-t)^{-1} 1_{t \le \tau}$.

Contents

Definition

The fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ is a centered Gaussian process $B^H = \{B^H_t, t \geq 0\}$ with stationary increments and the covariance function

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For $H > 1/2$ (the case we consider here) fBm has long memory.

B^H is almost surely Hölder continuous with any exponent $\gamma < H$.

Integration

For $\alpha \in (0,1)$ fractional derivatives

$$\big(D^{\alpha}_{a+} f\big)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(u)}{(x-u)^{1+\alpha}}du\right)1_{(a,b)}(x),$$

$$\big(D^{1-\alpha}_{b-} g\big)(x) = \frac{e^{i\pi\alpha}}{\Gamma(\alpha)}\left(\frac{g(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_x^b \frac{g(x)-g(u)}{(u-x)^{2-\alpha}}du\right)1_{(a,b)}(x).$$

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Assume that $D^{\alpha}_{a+}f \in L_p[a,b],\ D^{1-\alpha}_{b-}g_{b-} \in L_q[a,b]$ for some $p \in (1, 1/\alpha)$, $q = p/(p-1)$, where $g_{b-}(x) = g(x) - g(b)$.

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We can define

$$\int_a^b f(x)dg(x) = e^{-i\pi\alpha}\int_a^b \big(D^{\alpha}_{a+} f\big)(x)\big(D^{1-\alpha}_{b-} g_{b-}\big)(x)dx.$$

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$$\int_a^b f(x) dg(x) = e^{-i\pi\alpha} \int_a^b \left(D_{a+}^{\alpha} f\right)(x) \left(D_{b-}^{1-\alpha} g_{b-}\right)(x) dx.$$

If $f \in C^{\mu}[a,b]$, $g \in C^{\nu}[a,b]$ with $\mu + \nu > 1$, then $\int_a^b f(x) dg(x)$ is a limit of integral sums.

For any $\alpha \in (1-H, 1)$, $D_{b-}^{1-\alpha} B_{b-}^{H} \in L_\infty[a,b]$, so we can define for f with $D_{a+}^{\alpha} f \in L_1[a,b]$

$$\int_a^b f_s dB_s^H = e^{-i\pi\alpha} \int_a^b \left(D_{a+}^{\alpha} f\right)(x)\left(D_{b-}^{1-\alpha} B_{b-}^{H}\right)(x) dx.$$

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$$\int_a^b f_s dB^H_s = e^{-i\pi\alpha}\int_a^b \left(D^\alpha_{a+}f\right)(x)\left(D^{1-\alpha}_{b-}B^H_{b-}\right)(x)dx.$$

Consider the following norm for $\alpha \in (1-H,1/2)$:

$$\|f\|_{1,\alpha,[a,b]} = \int_a^b \left(\frac{|f(s)|}{(s-a)^\alpha} + \int_a^s \frac{|f(s)-f(z)|}{(s-z)^{1+\alpha}}dz\right)ds.$$

For simplicity we will abbreviate $\|\cdot\|_{\alpha,t} = \|\cdot\|_{1,\alpha,[0,t]}$.

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For simplicity we will abbreviate $\|\cdot\|_{\alpha,t} = \|\cdot\|_{1,\alpha,[0,t]}$.

Theorem (Azmoodeh, Mishura, Valkeila (2011))

Let $f : \mathbb{R} \to \mathbb{R}$ be a function of locally bounded variation, $F(x) = \int_0^x f(y)dy$. Then for any $\alpha \in (1-H, 1/2)$ $\|f(B_\cdot^H)\|_{\alpha,1} < \infty$ a.s. and

$$F(B_t^H) = \int_0^t f(B_s^H) dB_s^H.$$

For $F(x) = |x|$:

$$|B_t^H| = \int_0^t \operatorname{sign} B_s^H dB_s^H.$$

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Fact: for $0 < s \le t \le 1$ $\mathrm{P}(B_s^H B_t^H < 0) \le C(t-s)^H t^{-H}$.

$$\begin{aligned}
\mathrm{E}\left[\,\|\operatorname{sign} B^H\|_{\alpha,t}\right] &= \mathrm{E}\left[\int_0^t \left(\frac{\left|\operatorname{sign} B_s^H\right|}{s^\alpha} + \int_0^s \frac{\left|\operatorname{sign} B_s^H - \operatorname{sign} B_z^H\right|}{(s-z)^{1+\alpha}} dz\, ds\right)\right] \\
&\le C + \int_0^t \int_0^s \frac{\mathrm{E}\left[\,\left|\operatorname{sign} B_s^H - \operatorname{sign} B_z^H\right|\,\right]}{(s-z)^{1+\alpha}} dz\, ds \\
&= C + 2\int_0^t \int_0^s \frac{P(B_s^H B_z^H < 0)}{(s-z)^{1+\alpha}} dz\, ds \\
&\le C + C\int_0^t \int_0^s (s-z)^{H-1-\alpha} s^{-H} dz\, ds < \infty.
\end{aligned}$$

Contents

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space endowed with a P-complete *left-continuous* filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0,1]\}$, and B^H be $\mathbb{F}$-adapted fractional Brownian motion.

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Lemma

There exists an $\mathbb{F}$-adapted process $\varphi = \{\varphi_t, t \in [0,1]\}$ such that

- *For any $t < 1$ and $\alpha \in (1-H, 1/2)$ $\|\varphi\|_{\alpha,t} < \infty$ a.s., so integral $v_t = \int_0^t \varphi_s dB_s^H$ exists as a generalized Lebesgue–Stieltjes integral.*
- $\lim_{t\to 1-} v_t = \infty$ *a.s.*

Key ingredient of the proof is small ball estimate for fBm:

$$P\left(\sup_{t\in[0,T]} |B_t^H| < \epsilon\right) \le e^{-cT\epsilon^{-1/H}} \text{ for } \epsilon \le T^H.$$

Proof

Fix arbitrary $\gamma \in (1, 1/H)$ and $\beta \in (0, \frac{1}{\gamma H} - 1)$. Let $\Delta_n = n^{-\gamma}/\zeta(\gamma)$, $\zeta(\gamma) = \sum_{n\geq 1} n^{-\gamma}$,$t_n = \sum_{k=1}^{n} \Delta_k$, $n \geq 0$, we have $t_n \to 1-$, $n \to \infty$. Denote also $f_\beta(x) = (1+\beta)x^\beta \operatorname{sign} x$, so that $\int_0^x f_\beta(z)dz = |x|^{1+\beta}$, $x \in \mathbb{R}$.

Let $\tau_n = \min\left\{t \geq t_{n-1} : \left|B_t^H - B_{t_{n-1}}^H\right| \geq n^{-1/(1+\beta)}\right\} \wedge t_n$ and define

$$\varphi_t = \sum_{n=1}^{\infty} f_\beta(B_t^H - B_{t_n}^H)\mathbb{1}_{[t_{n-1},\tau_n)}(t).$$

Estimate $\|\varphi\|_{\alpha,t} < \infty$ is easy.

By the Itô formula, for $t \in [t_{n-1}, t_n)$

$$v_t = \int_0^t \varphi_s dB_s^H = \sum_{k=1}^{n-1} \left|\Delta B_k^H\right|^{1+\beta} + \left|B_{t\wedge\tau_n}^H - B_{t_{n-1}}^H\right|^{1+\beta},$$

where $\Delta B_k^H = B_{\tau_k}^H - B_{t_{k-1}}^H$, $k \geq 1$. We have $v_t \geq v_{t_n}$ for $t \geq t_n$, so it is enough to show that $v_{t_n} \to \infty$, equivalently, $\sum_{n=1}^{\infty} \left|\Delta B_n^H\right|^{1+\beta} = \infty$.

Observe that $\left|\Delta B_n^H\right|^{1+\beta} \geq 1/n$ provided that $\tau_n < t_n$. Therefore, defining $A_n = \left\{\sup_{t\in[t_{n-1},t_n]} \left|B_t^H - B_{t_{n-1}}^H\right| < n^{-1/(1+\beta)}\right\}$, $n \geq 1$, it is enough to show that almost surely only finite number of the events A_n happens. Using the small ball estimate and stationarity of increments of B^H, we obtain

$$\mathrm{P}(A_n) = \mathrm{P}\left(\sup_{t\in[0,\Delta_n]} \left|B_t^H\right| < n^{-1/(1+\beta)}\right) \leq \exp\left\{-c\zeta(\gamma)^{-1} n^{-\gamma+\frac{1}{H(1+\beta)}}\right\},$$

so $\sum_{n\geq 1} P(A_n) < \infty$ since $\frac{1}{H(1+\beta)} > \gamma$. Thus, we get the desired statement from the Borel-Cantelli lemma.

Theorem

For any distribution function G there exists an adapted process ζ such that $\|\zeta\|_{\alpha,1} < \infty$ and the distribution function of $\int_0^1 \zeta_s dB_s^H$ is G.

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Proof. Take a monotone function $g : \mathbb{R} \to \mathbb{R}$ such that $g(B^H_{1/2})$ has distribution G. Let φ be the process constructed in lemma, $v_t = \int_{1/2}^t \varphi_s dB_s^H$. Define $\tau = \min\left\{t \geq 1/2 : v_t = |g(B^H_{1/2})|\right\}$. Since $v_t \to \infty$ as $t \to 1-$ a.s., we have $\tau < 1$ a.s. Now put

$$\zeta_t = \varphi_t \operatorname{sign} g(B^H_{1/2}) \mathbf{1}_{[1/2,\tau]}(t).$$

Theorem

For any $\mathcal{F}_1$-measurable variable ξ there exists an $\mathbb{F}$-adapted process ψ such that

- *For any $t < 1$ and $\alpha \in (1 - H, 1/2)$ $\|\psi\|_{\alpha,t} < \infty$ a.s.*
- $\lim_{t\to 1} \int_0^t \psi_s dB_s^H = \xi$ *a.s.*

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Proof. $z_t = \tan \mathbb{E}[\arctan \xi | \mathcal{F}_t]$ is $\mathbb{F}$-adapted and $z_t \to \xi$, $t \to 1-$.

Let $\{t_n, n \geq 1\}$ be arbitrary increasing sequence of points from $[0, 1]$ converging to 1.

By Lemma, there exists an $\mathbb{F}$-adapted process φ^n on $[t_n, t_{n+1}]$ such that $v_t^n = \int_{t_n}^t \varphi_s^n dB_s^H \to +\infty$, $t \to t_{n+1}-$.

Now denote $\xi_n = z_{t_n}$ and $\delta_n = \xi_n - \xi_{n-1}$, $n \geq 2$, $\delta_1 = \xi_1$. Take $\tau_n = \min\{t \geq t_n : v_t^n = |\delta_n|\}$ and define

$$\psi_t = \sum_{n\geq 1} \varphi_t^n \mathbb{1}_{[t_n,\tau_n]}(t) \operatorname{sign} \delta_n.$$

Main theorem

Theorem

Let for a random variable ξ there exist an $\mathbb{F}$-adapted almost surely a-Hölder continuous process $\{z_t, t \in [0,1]\}$ such that $z_1 = \xi$. Then for any $\alpha \in (1-H, (1-H+a) \wedge 1/2)$ there exists an $\mathbb{F}$-adapted process ψ such that $\|\psi\|_{\alpha,1} < \infty$ and $\int_0^1 \psi_s dB_s^H = \xi$.

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(Blue) assumption is equivalent to: there exist $a > 0$, sequence $\{t_n, n \geq 1\}$, $t_n \uparrow 1$, sequence of rv's $\{\xi_n, n \geq 1\}$ such that ξ_n is $\mathcal{F}_{t_n}$-measurable and $|\xi_n - \xi| = O(|t_n - 1|^a), \ n \to \infty$.

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Theorem

Let ξ be an $\mathcal{F}_1$-measurable random variable and let there exist an $\mathbb{F}$-adapted continuous process ψ such that for some $\alpha > 1 - H$ $\|\psi\|_{\alpha,1} < \infty$ a.s. and $\int_0^1 \psi_s dB_s^H = \xi$. Then the assumption of main theorem is satisfied.

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Example

$\xi = F(B^H_{s_1}, \ldots, B^H_{s_n})$, where $F : \mathbb{R}^n \to \mathbb{R}$ is locally Hölder continuous with respect to each variable. (Set $z_t = F(B^H_{s_1 \wedge t}, \ldots, B^H_{s_n \wedge t})$.)

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Example

$\xi = G(\{B^H_s, s \in [0,1]\})$, where $G\colon C[0,1] \to \mathbb{R}$ is locally Hölder continuous with respect to the supremum norm on $C[0,1]$. In the case one can set $z_t = G(\{B^H_{s \wedge t}, s \in [0,1]\})$.

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Example

Assume that $\mathbb{F} = \{\mathcal{F}_t = \sigma(B^H_s, s \in [0,t]), t \in [0,1]\}$. It is well known that there exists a Wiener process W such that its natural filtration coincides with $\mathbb{F}$. Define $\xi = \int_{1/2}^1 g(t) dW_t$, where $g(t) = (1-t)^{-1/2} \left|\log(1-t)\right|^{-1}$. Then ξ does *not* satisfy the main theorem assumption.

Sketch of the proof

Choose special γ, κ, let $\Delta_n = n^{-\gamma}/\zeta(\gamma)$, $t_n = \sum_{k=1}^{n} \Delta_k$.

Denote $\xi_n = z_{t_n}$, $\delta_n = |\xi_n - \xi_{n-1}|$.

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Define

$$\tau_n = \min\left\{t \geq t_{n-1} : n^\kappa \left|B_t^H - B_{t_{n-1}}^H\right| = \delta_{n-1}\right\} \wedge t_n$$

and set

$$\psi_t = n^\kappa \operatorname{sign}(B_t^H - B_{t_{n-1}}^H) \operatorname{sign}(\xi_{n-1} - \xi_{n-2}) \mathbb{1}_{t \leq \tau_n}$$

for $t \in [t_{n-1}, t_n)$. By the Itô formula,

$$v_{t_n} = v_{t_{n-1}} + n^\kappa \left|B_{\tau_n}^H - B_{t_{n-1}}^H\right| \operatorname{sign}(\xi_{n-1} - v_{t_{n-1}}),$$

so we have $v_{t_n} = \xi_{n-1}$ provided $\tau_n < t_n$.

Sketch of the proof (cont'd)

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Sketch of the proof (cont'd)

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Let φ_t^n be an adapted process on $[t_{n-1}, t_n]$ such that $v_t^n := \int_{t_{n-1}}^{t_n} \varphi_s^n dB_s^H \to \infty$, $t \to t_n-$, define

$$\tau_n = \min\left\{t \geq t_{n-1} : v_t^n = \left|\xi_{n-1} - v_{t_{n-1}}\right|\right\}$$

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Next we argue using the Borel–Cantelli lemma and the small ball estimate that almost surely there is $N(\omega)$ such that $v_{t_n} = \xi_{n-1}$ for $n \geq N(\omega)$.

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Finally we show that $\|\psi\|_{\alpha,1} < \infty$, which together with the previous gives $\int_0^1 \psi_s dB_s^H = \xi$.

Contents

Consider a fractional (B, S)-market:

$$B_t = \exp\left\{\int_0^t r_s ds\right\}$$
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Definition

Portfolio is $\mathbb{F}$-adapted process $\Pi = (\Pi_t)_{t\in[0,1]} = (\pi_t^0, \pi_t^1)_{t\in[0,1]}$.
Value of portfolio Π at time t is

$$V_t^\Pi = \pi_t^0 B_t + \pi_t^1 S_t.$$

Portfolio is *self-financing* (SF) if

$$dV_t^\Pi = \pi_t^0 dB_t + \pi_t^1 dS_t.$$

Discounted value of a self-financing portfolio

$$C_t^\Pi = V_t^\Pi B_t^{-1}.$$

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$$dC_t^\Pi = \pi_t^1 dX_t,$$

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ξ is *weakly hedgeable* if there is SF portfolio Π (a *weak hedge*), s.t. $\lim_{t\to 1-} V_t^\Pi = \xi$ a.s.
Initial portfolio value V_0^Π is *hedging cost* (*weak hedging cost*).

Theorem

The fractional (B, S)-market model admits strong arbitrage.

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Any contingent claim ξ in the fractional (B, S)-market is weakly hedgeable. Moreover, its weak hedging cost can be any real number.

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Assume that for a contingent claim ξ there exists an $\mathbb{F}$-adapted almost surely Hölder continuous process $\{z_t, t \in [0, 1]\}$ with $z_1 = \xi$. Then ξ is hedgeable and its hedging cost can be any real number.