Fractional Brownian Motion in a (Coco)Nutshell

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Outline

Contents

F. Biagini, Y. Hu, B. Oksendal, T. Zhang
Stochastic Calculus for Fractional Brownian Motion and Applications
Probability and Its Applications, Springer, 2008.

Yu. Mishura
Stochastic Calculus for Fractional Brownian Motion and Related Processes
Lecture Notes in Mathematics **1929**, Springer, 2008.

I. Nourdin
Selected Aspects of Fractional Brownian Motion
Bocconi & Springer Series, Springer, 2012.

D. Nualart
The Malliavin calculus and related topics, 2nd Edition
Probability and Its Applications, Springer, 2006.

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The fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a centered Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with stationary increments and the covariance function

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$H = 1$: $B_t^H = \xi t$ with ξ standard Gaussian

Properties of increments

Covariance (Exercise)

$$\mathsf{E}\left[\left(B^H_{t_1}-B^H_{s_1}\right)\left(B^H_{t_2}-B^H_{s_2}\right)\right]=\frac{1}{2}\left(|t_2-s_1|^{2H}+|t_1-s_2|^{2H}-|t_1-s_1|^{2H}-|t_2-s_2|^{2H}\right).$$

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Dependence (Exercise)

If $s_1 < t_1 < s_2 < t_2$ (so the intervals $[s_1,t_1]$ and $[s_s,t_2]$ are non-intersecting), then the increments $B^H_{t_1}-B^H_{s_1}$ and $B^H_{t_2}-B^H_{s_2}$ are

- positively correlated for $H > 1/2$;

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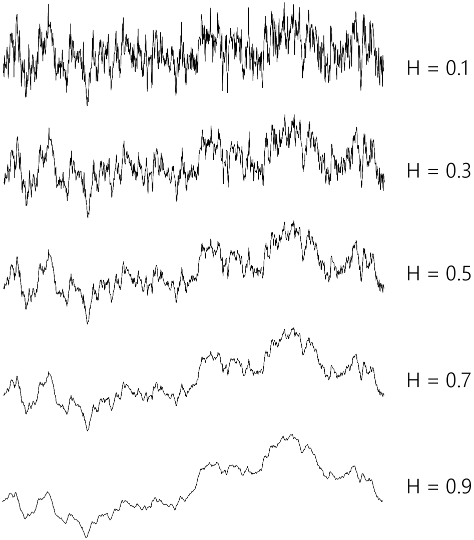
Hint: Use the convexity (concavity).

fBm paths



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Fractional discrete noise: the stationary sequence $\xi_n = B^H_{n+1} - B^H_n$.
Covariance:

$$\rho_H(n) = \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right), \ n \geq 1.$$

So $\sum_{n=1}^{\infty} \rho_H(n) = +\infty$ for $H > 1/2$ (the long-range dependence).

Continuity of fBm

The variance of increments is

$$\mathsf{E}\left[\left(B_t^H - B_s^H\right)^2\right] = |t-s|^{2H}.$$

Then it can be shown that fBm is γ-Hölder continuous:

$$\left|B_t^H - B_s^H\right| \le C(\omega)|t-s|^{\gamma}$$

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A nice way is the Garsia–Rodemich–Rumsey inequality:

For $f \in C([0,T])$ and $p > 0$, $\theta > 1/p$

$$\sup_{0\le v<u\le T} \frac{|f(u)-f(v)|}{(u-v)^{\theta-1/p}} \le C_{p,\theta,T}\left(\int_0^T\int_0^T \frac{|f(x)-f(y)|^p}{|x-y|^{\theta p+1}}\,dx\,dy\right)^{1/p}.$$

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Exercise

Deduce the Hölder continuity from the GRR inequality.

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Open problem

Let B^H be given by the above integral representation. Then B^H share the points of local extrema with its underlying Wiener process W.

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Note that this works for $H \in (0, 1/2)$ as well. There is also an inverse representation.

Contents

Coeurjolly, J.-F.
Simulation and identification of the fractional Brownian motion: a bibliographical and comparative study.
Journal of Statistical Software **5**, 1–53.

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$$V_{2,n} \overset{d}{=} T^{2H} n^{-2H} \sum_{k=1}^{n} \xi_k^2,$$

where $\{\xi_k, k \ge 1\}$ is a stationary standard Gaussian sequence with the covariance

$$\rho_H(n) = \mathsf{E}[\xi_1 \xi_{n+1}] = \rho_H(n) = \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H} \right), \ n \ge 1.$$

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Thanks to the ergodic theorem,

$$V_{2,n} \overset{d}{=} T^{2H} n^{1-2H} \frac{1}{n} \sum_{k=1}^{n} \xi_k^2 \sim T^{2H} n^{1-2H}, n\to\infty.$$

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To eliminate a possible drift, consider second order differences:

$$V'_{2,n} = \sum_{k=1}^{n-1}\left(B^H_{(k+1)T/n} + B^H_{(k-1)T/n} - 2B^H_{kT/n}\right)^2,$$

the asymptotic is the same up to a constant (exercise), although the variance is bigger.

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The situation is delicate:

For $H \in (0, 3/4)$, there is a usual central limit theorem:

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$$n^{2-2H}\left(n^{2H-1}V_{2,n} - T^{2H}\right) \to \zeta_H T^{2H}, n \to \infty,$$

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in law, where ζ_H has some very special "Rosenblatt distribution", *essentially* depending on H!

Confidence intervals

So one needs to consider alternatives, e.g. the realized cubic variation

$$V_{3,n} = \sum_{k=1}^{n} \left(B^H_{kT/n} - B^H_{(k-1)T/n} \right)^3 .$$

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Here

$$\sigma^2_{H,1} = \frac{1}{6} + \frac{1}{3} \sum_{m=1}^{\infty} \rho_H(m)^3$$

for $H \in (0, 1/2)$ and $\sigma^2_{H,1} = 3$ for $H \in (1/2, 1)$.

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$$n^{3H-1/2} V_{3,n} \to N(0, \sigma^2_{H,1} T^{6H});$$

for $H \in (1/2, 1)$,

$$n^{2H} V_{3,n} \to N(0, \sigma^2_{H,1} T^{6H}).$$

Here

$$\sigma^2_{H,1} = \frac{1}{6} + \frac{1}{3} \sum_{m=1}^{\infty} \rho_H(m)^3$$

for $H \in (0, 1/2)$ and $\sigma^2_{H,1} = 3$ for $H \in (1/2, 1)$. Again, the asymptotic variance depends on H, but one can plug in an estimator.

Exercise

Write the confidence intervals for H explicitly in terms of $V_{3,n}$.

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Reminder: fractional discrete noise is a stationary standard Gaussian sequence with the covariance

$$\rho_H(n) = \mathsf{E}[\xi_1 \xi_{n+1}] = \rho_H(n) = \frac{1}{2}\Big((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\Big), \ n \geq 1.$$

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So we have a centered Gaussian vector with known covariance matrix

$$\mathrm{Cov}(\xi) = \begin{pmatrix} 1 & \rho_H(1) & \rho_H(2) & \dots & \rho_H(N-2) & \rho_H(N-1) \\ \rho_H(1) & 1 & \rho_H(1) & \dots & \rho_H(N-3) & \rho_H(N-2) \\ \rho_H(2) & \rho_H(1) & 1 & \dots & \rho_H(N-4) & \rho_H(N-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_H(N-2) & \rho_H(N-3) & \rho_H(N-4) & \dots & 1 & \rho_H(1) \\ \rho_H(N-1) & \rho_H(N-2) & \rho_H(N-3) & \dots & \rho_H(1) & 1 \end{pmatrix}$$

We can obtain it by transforming linearly

$$(\xi_1, \xi_2, \dots, \xi_N)^\top = S \times (\zeta_1, \zeta_2, \dots, \zeta_N)^\top,$$

a vector $(\zeta_1, \zeta_2, \dots, \zeta_N)$ of independent standard Gaussians; S is an $N \times N$ matrix such that $SS^\top = \mathrm{Cov}(\xi)$.

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We assume that $N = 2^q + 1$ (for technical reasons).

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Idea: to extract the square root, *enlarge* the matrix $\mathrm{Cov}(\xi)$ by embedding it into a *circulant* matrix

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{M-2} & c_{M-1} \\ c_{M-1} & c_0 & c_1 & \dots & c_{M-3} & c_{M-2} \\ c_{M-2} & c_{M-1} & c_0 & \dots & c_{M-4} & c_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \dots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \dots & c_{M-1} & c_0 \end{pmatrix}$$

where $M = 2(N-1)$ and

$$c_0 = 1,$$

$$c_k = \begin{cases} \rho_H(k), & k = 1, 2, \dots, N-1, \\ \rho_H(M-k), & k = N, N+1, \dots, M-1. \end{cases}$$

The circulant matrix C is easily diagonalized:

$$C = Q\Lambda Q^{\top},$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_M)$, $Q = \left(q_{jk}\right)_{j,k=1}^{M}$ with

$$\lambda_k = \sum_{j=0}^{M-1} c_j \exp\left\{-i2\pi \frac{jk}{M}\right\},$$

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Exercise

Check that $C = Q\Lambda Q^*$.

Now finding square root is straightforward: $R = Q\Lambda^{1/2}Q^*$, where $\Lambda^{1/2} = \mathrm{diag}(\lambda_1^{1/2}, \dots, \lambda_M^{1/2})$.

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8. Take real part of $\xi_1, \dots, \xi_N$ to get fractional discrete noise
9. Multiply by $(T/N)^H$ to get increments of fBm;
10. Take cumulative sums to get fBm

Matlab code (guaranteed to work)

```
T = 3; H = 0.7; G = 2*H; % fBm with H = 0.7 on [0,3]
q = 20; N1 = 2^q; M = 2*N1; % about million values; N1 = N-1
rhoH = @(n) ((n+1).^G + abs(n-1).^G - 2.* n.^G)./2; % covariance
c = zeros(M,1); % initialize; Matlab counts starting from 1, so
g = rhoH((0:N1)'); h = flipdim(g(2:(N1+1)),1); % some mess here
c(1:(N1+1)) = g; c((N1+2):M); % and here with c_0,...,c_{M-1}
lambda = fft(c); % compute lambda
zeta = randn(M,1); % generate standard Gaussians
Qz = ifft(zeta); % compute Q^* zeta /sqrt(M)
xi = real(fft(Qz.*lambda.^0.5)); % compute xi
fbmincrements = (T/N1)^H .* xi(1:N1); % those are your increments
fbmpath = zeros(N1+1,1); % and here is your path
fbmpath(2:(N1+1)) = cumsum(fbmincrements); % starting from zero
```

Result